# Graph Theory

B. Math. II

Mid-Term Examination

**Instructions:** All questions carry ten marks. All graphs are assumed to be simple.

1. Define *connected component* of a graph. Prove that a graph $G$ with $n$ vertices and $e \leq n-1$ edges has $n-e$ connected components if and only if it contains no cycle.

2. A *cut-edge* of a graph $G$ is an edge such that its removal increases the number of connected components of $G$. Prove that a graph having no vertex of odd degree has no cut-edge. Further, for each $k \geq 1$, give an example of a $(2k+1)$-regular graph with at least one cut-edge.

3. Define a *Hamiltonian* graph. If $G$ is Hamiltonian and if $S \subseteq V(G)$, then prove that the induced graph on $V(G) \setminus S$ has at most $|S|$ connected components.

4. Define *connectivity* of a graph. Prove that a graph $G$ with at least three vertices is 2-connected if and only if given any two distinct vertices of $G$, there exist at least two internally disjoint paths between them.

5. Define a *planar* graph. Let $n \geq 3$ be a natural number and let $S$ be a subset of $n$ points in the plane such that the distance between any two distinct points of $S$ is at least one. Then, prove that there are at most $3n-6$ pairs $u, v$ in $S$ such that $d(u, v) = 1$.